IBN HALDUN UNIVERSITY
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MASTER THESIS

AN INTRODUCTION TO TRANSCENDENTAL EXPOSITION OF SKOLEM PARADOX

ÖMER FARUK ERDEMİR

THESIS SUPERVISOR: ASST. PROF. M. İKBAL BAKIR

ISTANBUL, 2020

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by

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A thesis submitted to the School of Graduate Studies in partial fulfillment of the requirements for the degree of Master of Arts in Philosophy

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APPROVAL PAGE

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Arts in Philosophy

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ÖZ

SKOLEM PARADOKSU’NUN TRANSANDANTAL TEŞHİRİNE GİRİŞ

Yazar Erdemir, Ömer Faruk
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Bu tezde Skolem paradoksunun transandantal yönden bir incelemesi gerçekleştirilmiştir. Skolem paradoksu sembolik mantık, aksiyomatik küme kuramı ve felsefeyi buluşturan bir evlektir ve bu alanlara dair gerçekci yaklaşımın temellerini sarsan bir paradoks olarak yorumlanmıştır. Bu yorumun geçerliliğini sorgulamak üzere bu tezde öncelikle bu yorumun dayandığı temel varsayımlar meydana çıkartılmıştır. Buna göre belirlenen ilk varsayım biçimsel dilin unsurları ile alakalıdır. Bu varsayıma göre ilgili unsurlar, bu dilin içerdiği sembollerin sadece listelenmesi ve sayılıp dökülmesi yardımıyla sentaksa dayalı olarak kurulmaktadırlar. Belirlenen ikinci varsayıma göre ise küme kuramsal kavramların doğru anlaşılmasının tek ve doğru yolu aksiyomatik yaklaşımdır. Kant’ın transandantal düşüncesi ekseninde bu iki varsayım sorgulanmıştır. Buna göre küme kuramsal bir kavram olan ve Skolem paradoksu tarafından nesnelliği ve mutlaklığı sorgulanan küme dünyası kavramının küme kuramsal söylem için zorunlu olduğu ifade edilmiştir. Kant’ın bakış açısından küme dünyası aklın şemaları üzerinden tesis edilen biçimsel bütünlüktür. Bunu tespit ettikten sonra bu biçimsel bütünlüğün tesisinde yer alan şemaların aynı zamanda Cantor’un köşegen kanıtlamasının nesnel temsili için gerekli olduğu ileri sürülmüştür. Böylece bu şemalar üzerinden küme dünyasının bir anlamda nesnelliği ve mutlaklığı haiz olduğu sonucuna ulaşılmıştır.

Bu inceleme sonucunda Lövenheim-Skolem teoreminin ve Skolem paradoksunun bir yorumu elde edilmiştir. Buna göre ilgili teoremde küme dünyasına dair iki bakış açısı rol oynamaktadır ve eğer bu iki bakış açısı dikkatle ayrıştırılırsa Skolem paradoksunun bahsi geçen alanlara dair gerçekçi bir yaklaşıma tehdit oluşturamayacağı ifade edilmiştir.

Anahtar Kelimeler: Matematiksel Mantık, Model Teorisi, Paradoks, Kant, Cantor, Skolem

ABSTRACT

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In the present thesis, we carry out a transcendental investigation of Skolem paradox. Skolem paradox is a paradox at the junction point of symbolic logic, axiomatic set theory and philosophy. Skolem paradox is generally understood as a model theoretic paradox which undermines the realist approach to these areas. In order to show why this paradox doesn't pose a threat to the realist approach to these areas, we begin with clarifying the crucial assumptions on which such an understanding relies. According to our investigation, first of these assumptions is that the elements of formal language is constituted just by listing and enumerating its symbols with the help of recursive functions defined in syntactical manner. Second of these assumptions is that the axiomatic approach is only way to understand the set theoretic notions such as the universe of sets and uncountability. As for the second assumption we argue, in relation to Kant's transcendental thought, that cognition of the universe of sets is necessary to the set theoretic discourse. From Kantian point of view the universe of sets is the formal whole of reason and we determine the schemata of reason on which constitution of the formal whole depends. As for the first assumption we claim that cognition and constitution of the representation of Cantor's diagonal argument depend on the same schemata of reason. We conclude that these schemata provide in a manner objectivity and absoluteness to the universe of sets. In view of our investigation we obtain an interpretation of Skolem paradox in relation to Lövenheim-Skolem Theorem. According to this interpretation Lövenheim-Skolem Theorem implies two different points of view about the universe of sets and if these points of view are differentiated Skolem paradox doesn't pose a threat to the realist approach.

Keywords: Mathematical Logic, Model Theory, Paradox, Kant, Cantor, Skolem

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LIST OF SYMBOLS AND ABBREVIATIONS

LST	Lövenheim-Skolem Theorem
SP	Skolem Paradox
ZFC	Zermelo-Fraenkel Set Theory Including Axiom of Choice
Wff	Well-Formed Formula
α, β	Metalinguistic Variable
$\mathbb{N}$	Set of Natural Numbers
$\mathbb{Q}$	Set of Rational Numbers
$\mathbb{R}$	Set of Real Numbers
$\exists$	Existential Quantifier
$\wedge$	Conjunction
$\vee$	Disjunction
$\neg$	Negation
$\rightarrow$	Conditional
$\in$	Set Membership
$\leq$	Less than or Equal to
$<$	Less than
$=$	Equality
$\neq$	Inequality
$+$	Plus

CHAPTER I

A LOGICO-HISTORICAL INTRODUCTION

1.1. Notion of Set and Set Theory

A thing is any object of thought. Thing is given in judgement and determined through all that can be judged about it. Different things can be the object of the same judgement and thought from a common point of view. From this common point of view these things form a collection (Dedekind 1963, p.44-45). Broadly speaking, this collection itself is called set and these things are called elements or members of that collection. This broad conception of set is seen in the first explicit definition of set of Cantor who is considered to be the father of set theory: "By a manifold [mannigfaltigkeit] or a set [menge] I understand in general every Many that can be thought of as One, i.e., every collection of determinate elements which can be bound up into whole through a law, and with this I believe to define something that is akin to the Platonic εἶδος or Ιδέα (Ferreiros 2007, p.265)." In other words different things (determinate elements, Many) can be collected so as to form whole, One, through a property or a law, i.e. a common point of view. For example, natural numbers such as 2, 4, 6, 8 can be collected in set (whole, one) through following a property or a law: every even natural number that is less than 10. By doing so, these numbers are collected in the set of x such that x is even and x is less than 10.

As it can be seen the concept of set may be considered as self evident. But it is defined in set theory through axioms for reasons which we investigate below. Axioms of set theory are relatively simple properties of sets formulated formally by mathematicians like Ernst Zermelo and Abraham Fraenkel and all the other properties of sets follow logically from these axioms (Hrbacek & Jech 1999, p.3) and hence proved as theorems. So, as for a strict conception of it, set is a mathematical object that satisfies the conditions put by the formal axioms. Since notion of set isn't self-evident, before the formulation of these axioms it may not be possible to claim that there is an object such as set and this object has certain porperties. Thus existence and basic properties of set is postulated by the formal axioms (Bagaria 2019) and constituted in the formal language called set theory.

Set theory has two different aspects. Firstly set theory is the mathematical discipline about abstract sets and their properties (Bagaria 2008, p.616). Basically, set theory is a theory which is obtained from predicate logic with addition of membership property. Apart from membership property “all other set-theoretic properties can be stated in in terms of membership with the help of logical means: identity, logical connectives and quantifiers (Hrbacek & Jech 1999, p.4).” Secondly because of the fact that almost all mathematical objects may be considered as sets (Suppes 1972, p.1) and formalized in the language of set theory (Bagaria 2019) it is regarded as the foundation of mathematics. This means that almost every mathematical object and theorem can be logically deduced from the totality of membership property, logical means and axioms of set theory which are stated in terms of them. For example the study about sets which are finite and have only finite sets as elements and their properties is formally equivalent to arithmetic (Bagaria 2019). This leaves the infinite sets to the core of set theory considered as separate mathematical discipline and “therefore it can be defined as the mathematical theory of the actual – as opposed to potential – infite (Bagaria 2019).”

In fact the set theory as a distinct mathematical discipline emerged after Cantor’s investigation of infinite series and related topics of analysis (Suppes 1972, p.2). Specifically it may be said that Cantor’s discovery that the set of real numbers cannot be put in a one-to-one correspondance with the set of natural numbers in 1873 marks the emergence of set theory as a separate mathematical discipline (Bagaria 2019). But, before Cantor’s works, set-theoretic mathematics are discussed in relation to the problem of characterization of integers, rational numbers and real numbers in the works of Dedekind and the problems of trigonometric series and topology in the works of Riemann (Ferreiros 2016). Conceptual and abstract approach by above mentioned mathematicans and many others to these and other problems gave rise to the set theory (Ferreiros 2007, p.xxiv). This also gives a historical hint why the set theory considered as the foundation of mathematics.

As it is said, set theory emerged with the discovery of Cantor which is about non-bijectablity of natural numbers and real numbers. In order to understand this discovery it is needed to answer the question about two sets having the same number of elements, i.e., the same size. According to Cantor two sets have the same cardinality, i.e., the same size if and only if they are bijectable, i.e., their elements can be put to one-to-one correspondence to each other (Bagaria 2019). The notion of one-to-one correspodance can be defined as the following: “A set A can be put into 1-1 correspondance with a set B if it is possible to match each element of A with one and only one element of B in a such way that no element of B is left out and no two distinct elements of A are matched with the same element of B (Smullyan & Fitting 1996, p.3).” It can be clearly seen that two finite sets have the same size if they have the same number of elements. To say that a finite set has the n number of elements is to put elements of that set into 1-1 correspondance with the set of positive whole numbers from 1 to n (Smullyan & Fitting 1996, p.3). The set of fingers on my left hand and the set of pencils on my desk have the same size if each set can be put into 1-1 correspondance with the set of positive whole numbers from 1 to 5. When Cantor applied the notion of cardinality to inifinite sets and questioned if all infinite sets have the same size or not, he made the discovery mentioned above (Smullyan & Fitting 1996, p.4).

Following Dedekind, infinite set can be defined as the following: A set is infinite if it is possible to put it into 1-1 correspondance with the proper subset of it (Ferreiros 2007, p.109). If it is not possible then the set is finite. The set of natural numbers can be put 1-1 correspondance with the set of even natural numbers. Thus it is infinite. Regarding 1-1 correspondance of infinite sets, Cantor defined two terms: countable and uncountable, i.e. respectively, denumerable and non-denumerable. “A countable set is one that can be put into one-to-one correspondence with the natural numbers. In other words, it is a set that we can enumerate, assigning a different natural number to each of its elements (Bagaria 2008, p.616).” So Cantor’s question can be rephrased as following: are all infite sets countable or do uncountable sets exist. He showed that the set of rationale numbers is countable. But, by using diagonel method, He proved that the set of reel numbers is uncountable (Bagaria 2008, p.616).[1] That is to say: if we try to put each element of the set of $\mathbb{R}$ 1-1 correpondance with one and only one element of the set of $\mathbb{N}$ in a way that no element of the set of $\mathbb{N}$ is left out and no two distinct elements of the set of $\mathbb{R}$ are matched with the same element of the set of $\mathbb{N}$ we will use up all elements of $\mathbb{N}$ before we have enumerated all elements of $\mathbb{R}$. Thus all infinite sets don't have the same size, i.e., there are countable and uncountable sets. This means that there are at least two different kinds of infinite sets.

After showing infinite sets having different sizes, Cantor investigated these sets from the ordinal point of view. From the cardinal point of view a number shows how many elements any set has. From the ordinal point of view a number shows the order of any set or element in the list in which they are counted. Let $\aleph_0$ denote the cardinality of natural numbers. Cantor considered this number as the first infinite ordinal denoted by ω (Bagaria 2008, p.618). Then Cantor showed that the cardinality of power set of $\mathbb{N}$, namely $2^{\aleph_0}$, is equal to the cardinality of $\mathbb{R}$, denoted by C. In other words $2^{\aleph_0} = \mathrm{C}$. In 1878 Cantor formulated famous Continuum Hypothesis and claimed that this is the second infinite ordinal denoted by ω_1 (Bagaria 2008, p.618). Continuing this reasoning 2^{C} is equal to third of them and so on. Thus infinite hierarchy of infinite sets gives us the universe of all sets, Cantor's paradise, in which there are sets of objects, sets of sets, sets of such sets and so on.

1.2. Interplay between Set Theory and Logic

It is possible to construct the universe of all sets beginning from things and successively applying "the set of" operation. It is also possible to construct it beginning "from nothing, i.e., the empty set, by successively applying "the set of" operation (Bagaria 2008, p.619)." Either way, following Dedekind, all that is judged, thought or affirmed about is an object considered as a collection or part of it. Considering any object in the hierarchy of objects brings to mind the tendency of traditional logic. In fact, according to Ferreiros, "The abstract, extensional notion of set developed gradually out of the older idea of concept-extension (Ferreiros 2007, p.xxi)." In the hierarchy of concepts, called tree of Porphyry, a concept is constituted

[1] We investigate this proof in detail in the chapter 4.

by the concepts found above which correspond to its intension or meaning, namely the properties which it involves. Any concept also constitutes another concepts found below and these are called its extension or class of things which concept is predicated (Ferreiros 2007, pp.51-52). Intension of a concept is also called comprehension and defined in one of the most famous traditional logic book known as the Port-Royal Logic as following: “I call comprehension of an idea the attributes that it contains in itself and that cannot be removed without destroying the idea (Arnauld & Nicole 1996, p.39).” Extension of concepts is also called class and defined in the Port-Royal Logic as following: “I call the extension of an idea the subjects to which idea applies (Arnauld & Nicole 1996, p.40)” and these subjects are either concepts or individuals.While the idea of concept-extension is used to analyze the hierarchal totality of objects, particularly the notion of extension is also used to analyse relation between concepts, classify propositions in terms of quantity, justify both the conversion of propositions (Arnauld & Nicole 1996, p.129) and the syllogistic modes of deduction (Arnauld & Nicole 1996, pp.162-165). Thus, by using the idea of concept-extension, traditional logicians were able to describe the reality either beginning from the most general concepts or from the individuals. And also they were able to describe the logical space which contains every valid conversion and syllogism.

Because of the fact that notion of extension has a crucial role in traditional logic and notion of set is similar to that of extension, set is considered as belonging to the roots of logic and set theory as part of logic from about 1850 (Ferreiros 2001, p.464). The reason for this can be shown thusly: The copula “is” included in every proposition can have one of the following meanings: identity, membership and inclusion and relation of membership and inclusion is the core relations of set theory (Ferreiros 2001, p.464). Since the copula “is” in every aspect is part of logical study then the notion of set and set theory belong to logic.

Putting aside the question on the nature of relation between set theory and logic, today, set theoretic concepts, objects and principles is used extensively in metatheory of first order predicate logic and model theory. To understand roles of these in above mentioned theories some prerequisite definitions is needed. A formal logical system is constituted of syntax, interpretation of this syntax and deductive apparatus. Syntax of a system is determined by a set of symbols and a set of rules showing how to compose formulas from these symbols. An interpretation of syntax is an assignment of meannings to these symbols and formulas. Deductive apparatus is determined by defining certain formulas as axioms and rules of inference which allows deduction of other formulas, i.e., theorems from these formulas, i.e. axioms. The theory about interpretation of a system's syntax is called model theory and the theory about deductive apparatus is called proof theory. The theory in which a formal logical system is studied is called metatheory of that system (Yaqub 2015, pp.87-89; Button & Walsh, pp.7-9).

Defining syntax of a formal logical system if the smallest bits of syntax that are capable of meaning assignment are sentence letters, this system is called propositional logic. If they are predicate, variable and name then it is called predicate logic. Since in logic primary focus is on logical truth and consequence any syntax of logic has to have symbols or formulas that are appropriate for assignment of truth values, namely true and false, as their meanings. In propositional logic sentence letters are appropriate for this assignment. But, in predicate logic, above mentioned symbols aren't appropriate for it. So as to enable the assignment of truth values in this system some other meanings for these are needed and sets are perfect match for them (Sider 2010, pp.117-118).

For example, in order to assign a truth value to formula like "Fa" ("F" stands for a predicate and "a" stands for a name) a non-empty set called domain and interpretation function are defined. Interpretation function (I) assings to "a" a member of the domain and also to "F" an one-place relation over domain, a kind of set drawn from the domain. If the domain is the set of philosophers then *I* might assign to "a" Immanuel Kant and to "F" the set of transcendental metaphysicians.Then we are able to say that "Fa" is true relative to this interpretation. In this case this interpretation is called 'model' for "Fa".

To conclude, generally speaking, the universe of sets is the mathematical representation of all possible objects and the world, i.e., all that there is and can be. In the universe of sets objects and world is thought as One and also as Whole. If the properties of its different elements and the order, in which they are given, are abstracted we have the cardinality of the universe. The cardinality gives us part of this mathematical unity and whole's nature. In particular the universe of sets gives us the universe through which formal objects (symbols, formulas and proofs) are regarded semantically according to logical truth and consequence. Thus set theory describes the part of logical space constructed from formal objects and set universe.

1.3. Paradoxes and Undecidability in Axiomatic Set Theory

Above mentioned conclusions such as regarding collections as objects, universe of sets as one, whole and notion of set as logically primitive is considered as false because of paradoxes found in set theory. Although these paradoxes have led negative results, they also have played role in the development of axiomatizations of set theory, the foundations of semantics and formal systems (Cantini & Bruni 2017). One of the paradoxes is discovered by Cantor and "realized that it contradicted the usual conception of sets as concept-extension (i.e., the principle of comprehension) (Ferreiros 2007, p.290)." This paradox is related to the notion of cardinality and implies that it'snt possible to think the universe of all sets consistently, i.e. as set. If the universe is a set then it has the maximum cardinality since all sets are the members of it. But it is possible to think of all subsets of this universe. By Cantor's theorem its cardinality is bigger then the cardinality of the universe. Thus there is a contradiction (Cantini & Bruni 2017; Ferreiros 2007, pp.291-294).

As a consequence, Cantor differentiated between collections that cannot be thought as unity, whole and completed, and collections can be thought as such. According to Cantor while sets are kind of latter collections, former collections represents the Absolute which "'can only be acknowledged, but never known, not even approximately known.' that is to say, it cannot be mathematically determined (Ferreiros 2007, p.291)." Thus the universe of all sets represents the Absolute and cannot be thought as thing or object.

Another one of the paradoxes is called Russell's paradox. This paradox is related to the idea that any well-defined property or law forms a set. This idea is formalized in set theory as the Comprehension Axiom. For example, if one defines a property such as being a student of Yalçın Koç, by this axiom, there exists the set whose elements are exactly those objects that satisfy the property. Russell discovered that from this axiom a contradiction follows. The paradox can be shown thusly: Let R be the set of all those sets which are not elements of themselves. The set of students of Yalçın Koç isn't a student of him thus it isn't element of itself. So this set is element of R. The set of abstract objects is an abstract object thus it is element of itself. So this set isn't element of R. What about R? Is R an element of itself? if it is, then it must satisfy the property of not being element of itself and so it isn't. If it isn't then it must not satisfy the property of not being element of itself and so it is. Thus there is a contradiction (Irvine & Deutsch 2016).

As a consequence of this paradox the conception of the notion of set and membership relation as self-evident and logically primitive is regarded as false and according to Russell, rethinking of foundations of both logic and mathematics was needed. Eventually the rethinking in the 'light' of paradoxes has led the formal axiom systems in mathematics (Ferreiros 2007, p.311).

Another conclusion is that there are properties which do not define and form sets. So any axiomatic set theory has to determine which properties define sets and which properties don't define. Any axiomatic set theory is supposed to include all former kind of properties and exclude the latters. But to do that, from the axioms of any set theory all and only truths about sets must be derived. This idea refers to the completeness theorem which can be stated thusly: In a formal theory about an object if a property is true of that object then it must be theorem in that theory, i.e. provable from the axioms. But according to Incompleteness Theorems proved by Kurt Gödel in 1931, it isn't possible (Hrbacek & Jech 1999, p.3). Thus there is at least one property which is true of sets and it cannot be proved in the formal theory about sets.

Truth or falsity of any logical and mathematical statement can be shown by deriving, respectively, it or negation of it from basic principles or axioms. Before Gödel's publication of theorems, the idea was to establish sufficent axioms and proof theory, which are represented in a rigorous symbolic notation, for deriving either truth or falsity for any logical and mathematical statement (Çitil 1994, p.1). But these theorems showed that such an idea isn't possible. The first completeness theorem says that in a formal axiomatic system, which is consistent and rich enough to represent truths about arithmetic, there is a statement that neither it nor negation of it provable. More strikingly according to second completeness theorem neither the formal statement, which asserts that such a rich formal axiomatic system is consistent, nor its negation can be proved in that theory. These are also true of any axiomatic set theory (Bagaria 2008, p.623).

There is another theorem of Gödel, called the completeness theorem for first order logic. According to this, axiomatic set theory is consistent if and only if it has a model. Thus reading this theorem and second incompleteness theorem togehter the result is that if the theory is consistent then it is not possible to prove in the theory that there is a model of it. Otherwise it would prove its own consistency (Bagaria 2008, p.623).

If there is a model of axiomatic set theory in which the statement is true and also there is a model in which negation of it true then this statement is called undecidable in that theory. Proof of consistency or undecidability of such statement is always relative. First one assumes that the theory is consistent thus it has a model. Secondly one constructs a model of that theory in which such statement is true. Thus the statement is true relative to that model. One of such statements is Continuum Hypothesis. Cantor's continuum hypothesis is undecidable in the axiomatic set theory abbreviated as ZFC, namely Zermelo-Fraenkel set theory including the axiom of choice. In 1938 Gödel constructed a model of ZFC where continuum hypothesis is true and in 1963 Paul Cohen, by using forcing method, found a way to show that there is a model of ZFC where continuum hyhothesis is false (Bagaria 2008, p.624).

1.4. Skolem Paradox

One of such paradoxes that have philosophically important consequences for both logic and set theory is the Skolem's paradox. Our focus in this thesis will be on this paradox that follows from Lövenheim-Skolem Theorem. In 1915 Lövenheim proved that if a first-order sentence has a model, then it has a countable model. In 1920 Skolem simplified Lövenheim theorem and generalized it. This version of theorem says that if a first-order theory (namely denumerable infinite set of sentences as

Skolem put it) has infinite models, then it has a countable model (Bays 2014; van Heijenoort 1967, pp.252-254).

This generalized version of the Lövenheim theorem is called the Lövenheim-Skolem Theorem. In 1922 Skolem put forth that from this theorem the paradox follows. Skolem paradox arises when we consider Cantor's theorem which proves that there are uncountable sets and also that the Cantorian set theory can be formulated as set of first-order sentences. Thus the set theory in which the existence of uncountable sets is proved can be satisfied by a model that is only countable, regarding that it has a model (Bays 2014). In this way, also, a countable model is made to satisfy the first-order sentence says that uncountable sets exits. When this sentence is interpreted ranging over an uncountable model, what the sentence is understood as saying coincides with the domain over which the sentece is interpreted. But when this sentence is interpreted over a countable model which satisfies it according to the Lövenheim-Skolem theorem, the question arises: how can a countable domain satisfy the sentence which is understood as saying that uncountable sets exits. This is the Skolem's paradox.

Firstly it should be noted that the Skolem's paradox isn't a paradox in the true sense of the word. It doesnt lead to a outright mathematical contradiction. "It is not a paradox in the sense of a antinomy; it is a novel and unexpected feature of formal systems (van Heijenoort 1967, pp.290-291)." Secondly while it is only apparent 'paradox', it is regarded as philosophically important. According to Skolem, from this theorem and paradox these philosophical consequences for the axiomatic set theory, among others, can be entailed: (1) relativity of set theoretic notions (Skolem 1922, p.296), (2) holding of the theorems of set theory in merely verbal sense (Skolem 1922, p.296), (3) inadequacy of set theory for ultimate foundation of mathematics (Skolem 1922, pp.300-301).

Now if we try to consider these consqeunces with above mentioned claims such as falsehood of regarding collections as objects, universe of sets as one, whole and notion of set as logically primitive following picture can be drawn before us:

(a) What a sentence, in particular a first-order formula of predicate symbolic logic, is understood as saying can only be decided by looking at the model over which the sentence is interpreted. Only then predicates can have a meaning. Outside of models there is no meaning of the predicates, particularly such as countable and uncountable (Boolos & Burgess & Jeffrey 2007, p.253).

(b) Since only by using sets one can decide truth of any first-order formula of predicate symbolic logic and truth is the other side of the coin of existence (in general sense), truth and exitence is ‘relative’. If set theory is made to range over countable domain, the theory will describe different reality from set theory ranging over uncountale domain. One can not differentiate these realities since a first-order set theory can not capture the notion of uncountability, countability and also intuitive notion of set (Bays 2014; Putnam 1980, p.466). In addition to this, outside the interpretations, these notions have no meaning.

In the present thesis we tackle philosophical consequences of Skolem paradox. For this, in the next chapter, we describe Lövenheim-Skolem theorem and Skolem paradox and show how it is understood by Skolem and mention others. In chapter 3, we deepen this inquiry via connecting the universe of sets to Kant’s ‘World’ idea. In chapter 4, we get to the root of the Skolem Paradox and inquire Cantor’s diagonal argument. With this we criticize reception of Skolem paradox and regulative use of World idea and try to give meaing of Skolem paradox in light of this inquiry and criticism.

CHAPTER II

EXPOSITION OF THE SKOLEM PARADOX

We begin this chapter with description of Lövenheim-Skolem Theorem and Skolem Paradox[2]. Although it is helpful to formally show a proof of LST in order to understand philopshical bearings of LST and SP, it is not crucial for it (Arsenijevic 2012, pp.64). Therefore, firstly, we give directly its main formulation and represent their meaning. Then we describe various Skolem's philosophical claims and mention others. Lastly we represent our position.

2.1. Sketch of Lövenheim-Skolem Theorem and Skolem Paradox

Lövenheim-Skolem Theorem: If a set of first-order sentences, Σ, has a model, then it has a countable model.

Straight forward meaning of this theorem can be shown as following. Let M be a model of Σ which consists of a domain D whose cardinality is greater than $\aleph_0$ and relations R_1, R_2 and so on which are defined on D. Then there is a model of Σ, M^I, whose D^I has cardinality of $\aleph_0$ and which has $R^I{}_1$, $R^I{}_2$ and so on which are defined on D^I. Therefore for every sentence α of Σ whose extra-logical symbols are just the relation-symbols which refer to R_1, R_2 and so on, since it is true in M then it is also true in M^I (Arsenijevic 2012, p.65).

Skolem Paradox: If first-order axiomatization of set theory, for example ZFC, has a model, M, then by LST it has a countable model, M^I. Since the theorem "there is uncountable set", $\exists x Fx$, can be derived from ZFC there is a set, m, which satisfies this theorem, Fm, in a way that 'm' is a member of M. If 'Fm' is true in M, it is also true in M^I. But M^I is only countable. For all sets if it is a member of M^I then it is

[2] From now on, the abbreviation LST amounts to Lövenheim-Skolem Theorem and SP amounts to Skolem Paradox.

countable. ‘m’ is a member of M^I. Then from former interpretation ‘m’ is uncountable and from latter interpretation ‘m’ is countable (Bays 2000, pp.4-5).

It should be noted that since LST is valid for any first-order axiomatization of set theory SP can show up in them. That is to say: “we can’t resolve the paradox by simply choosing a new axiomatization of set theory (or adding some new axioms to the axiomatization that we’re already using) (Bays 2014).” Also, at the risk of repeating, SP does not constitute mathematical contradiction for any such axiomatization.

2.2. Philosophical Bearings of Skolem Paradox: Skolem’s Position

As it is mentioned above the notion of Set isn’t regarded as self-evident. The reason for this can be enumerated thusly: (1) from mathematical perspective the notion of set in its original understanding, regarded as self-evident in the same sense as logical notions or in some other sense, is shown to lead paradoxes. (2) Also from wider and philosophical perspective the reason for this regard is Kant’s claim that there is no intellectual intuiton by which human can receive objects. Human’s only receptivity is sensuous (Kant 1998, pp.121-122). So any claim that there are some non-sensous objects and Human can know them prior to experience without appealing to any kind of sensibility (pure or empirical intuiton which belongs to sensilibity) is regarded by Kant as groundless.

Ever since Kant made this claim, both mathematicians and philosophers, because of their lack of conviction to Kant’s suggestion, have tried to find the source for mathematical knowledge (Coffa 1991, p.7). On the way of this investigation they run into some paradoxes and in the light of these some suggested that the only source of mathematical knowledge is axiomatization (Skolem 1922, p.291). On this conception, as far as the set theory concerned, the axioms of set theory don’t describe some given notions such as set, being a member but they implicitly define them. What set is the thing that satisfies these axioms. In other words “the axioms of set theory should not be seen as attempts at describing – or even partially describing – some antecendently given ‘intended model’ of set theory; instead, the intended *models* of set theory are simply those models which happen to satisfy our initial collection of set-theoretic axioms (Bays 2014).”

In principle one can hope that by being exposed to such a constructed set theory one could learn everything about set and its properties. In fact there are some set-theoretic concepts that one can learn by this way. Because models of an axiomatic set theory can capture and pin down them. For example empty set, singeleton, having twenty members are such concepts. Even with some preliminaries, models can capture infinity and finity. But one has to abandon the hope when it comes to uncountability. In the case of uncountability, although it is expected to capture, model theory can’t capture it as SP shows. This is the first place it “loses the ability to capture cardinality notions (Bays 2014).”

What set-theoretic concepts can be captured means can be explained as following. For this explanation, concepts of elemenatry equivalence and isomorphism are needed. These are two relationships between predicate logic interpretations.

Let Σ be set of first-order predicate logic sentences and M and M^I be models for Σ. M and M^I are elemantarily equivelant if it is the case that for every sentence of Σ α, α is true on M if and only if α is true on M^I. M and M^I are isomorphic if the case is such that: for each name and relation symbol that occurs in Σ there are referents in M and M^I and these referents can be put into one-to-one correpondance by isomorphism function (Yaqup 2015, p.167).

If M and M^I is isomorphic, components of them display similar relations and properties. So in the case of set theoretic concepts such as empty set, having twenty members and so on, models can display similar relations. Hence they can capture a sense of ‘empty set’ from within first-order axiomatic set theory (Bays 2014). But it isn’t the case when uncountability is considered. Although models in the case of uncountability are equivelant they are not isomorphic (Yaqub 2015, pp.178-179). They do not display similar relation and the relation of uncountability means differently, what ever it means, relative to a model. Hence there is no absolute meanig of uncountability all across models. This is the kind of relativity that Skolem talks about when he said “axiomatizing set theory leads to a relativity of set-theoretic notions, and this relativity is inseparably bound up with every thoroughgoing axiomatization (Skolem 1922, p.296)."

Skolem's argument can be summarized as the following: Regarding the notion of set and membership relation as self-evident leads to some paradoxes. The only way to describe and define these notions by avoiding these paradoxes is the axiomatization of set theory. SP shows that on such an axiomatization set theoretic notions are relative. Hence these notions are in fact relative (Bays 2014). Apart from uncountability notions such as finite, infinite, simply infinite sequence are also relative. We now try to summarize Skolem's first philosophical claim about relativity.

Before going further which role axiomatization plays for set theoretic notions should be noted. There is an epistemological role. According to this, axiomatization makes unclear and suspect set theoretic concepts intelligible and coherent. There is also an ontological role. According to this, axiomatization ensures the existence of such concepts. Skolem seems to accept both of these roles of axiomatization. Either way both roles as standing alone, if accepted, support relativity of such notions (McIntosh 1979, pp.325-326).

Skolem's second philosophical claim which is mentioned above is that "the theorems of set theory can be made to hold in a merely verbal sense (Skolem 1922, p.296)." Since these theorems consist of relative notions and these relative notions have no absolute meanings they do not have any object that they describe. If they had objects these would be collections. But in the case of ZFC, they are true because of the fact that some collections are not regarded as sets. Thus they hold only in verbal sense (Skolem 1922, p.296; McIntosh 1979, p.326).

Skolem's third philosophical claim is that set theory is inadequate for the ultimate foundation of mathematics. One of his arguments for this says that since arithmetic is certain while set theory isn't; latter can not be foundation of the former. Another argument relies on relativity mentioned above (McIntosh 1979, p.330). As we try to see in the sequel neither that SP is essential part of these arguments nor these arguments lead to the relativity (Bays 2014).

2.3. Philosophical Bearings of Skolem Paradox: The Skolemite Position

After Skolem presents his position, there has been an ongoing debate between philosophers and logicians who find Skolem's paradox and claims convincing or unconvincing. Michael D. Resnik called the position, who argues that LST and SP really show the relativity of uncountability and the absoluteness of countability, the Skolemite Position (Resnik 1966, p.425). In addition to Skolem's claims this position also argues that "no set theory is capable of producing genuine uncountable sets and that all these set are countable from an absolute point of view (Resnik 1966, p.426)." In addition to Skolem R. L. Goodstein, Hao Wang, William and Martha Kneale have advocated this position (Resnik 1966, p.426; Goodstein 1963; Wang 1964, p.565; Kneale & Kneale 1971, pp.711-712). Also in his reply to Resnik, William J. Thomas claimed this position (Thomas 1968, p.193). Clifton McIntosh also argued that LST entails weaker form of the relativity of set theoretic notions (McIntosh 1979, pp.314-315). And, of course, this list can be expanded.

The basic claims of the Skolemite Position can be presented as following: (1) The axiomatic set theory is only plausible approach to set theory. (2) The axiomatic set theory entails relativity of set theoretic notions. (3) Every set is countable from an absolute point of view (Bays 2014).

For the first step Skolemite argues that rejection of the axiomatic approach to set theory will lead to falling back on Platonistic approach which is unacceptable. Set theoretic paradoxes, other puzzles and the entire development of set theory favors the former approach. This shows why latter approach is unplausible (Bays 2014). For the second step Skomite follows the steps of Skolem. For the third step Skolemite argues that uncountability is always relative in a first-order set theory. On the other hand countability can be shown to be absolute even if countable domain satisfies the formula which is regarded to be saying "x is uncountable". Since while it is within the system appears to be uncountable from outside the system it is really countable. It appears to be uncountable only in the sense that there is no set in the system which can be put one-to-one correpondance with it while it is really countable outside the system, i.e., there is a set outside the system which can be put one-to-one correspondance with it (Resnik 1966, pp.426-427).

Several responses to the Skolemite argument can be enumareted as following. First of these responses is to try to clear away the mathematics that involved in LST and show that LST itself does not pose a problem for realistic approach to set theoretic notions (Bays 2014). By doing so Resnik investigates several ways of Skolemite argument and argues that these fail to prove the Skolemite position (Resnik 1966, pp.425-426). Paul Benacerraf argues that from LST no genuine paradox can be generated (Benacerraf 1985, p.101). Timothy Bays claims that "even on quite naive understandings of set theory and model theory, there is no such tension. Hence, Skolem's Paradox is not a genuine paradox, and there is very little reason to worry about (or even to investigate) the more extreme consequences that are supposed to follow from this paradox (Bays 2000, pp.1-2)."

Second of these responses critizes directly the axiomatic approach to set theory and defends instead the intuitive approach (Hart 1970, p.106). According to this, even for who is in the Skolemite position it is needed to accept some intuintive background theory so as to formulate model theoretic results. But this kind of response leaves open the special case of set theory (Bays 2014). Also according to this type of response, it has to be explained why one has to formulate the axiomatic set theory as first order but not the second order (Hart 1970, p.104). Since second order formulation of ZFC isn't subject to SP (Bays 2014).

Third of these reponses requires Skolemite to show how one "can identify sets accross differents models (Bays 2014)." For this Skolemite has to explain precisely what absolute point of view means and according to Resnik Skolemite will fail to do so (Resnik 1966, p.427).

Considering these arguments, Timothy Bays argues that Skolemite has to show that "Skolem's Paradox exposes a genuine tension between Cantor's theorem and the Löwenheim-Skolem theorem and that eliminating this tension requires a modification in our initial views about, e.g., set theory (Bays 2000, p.1)." According to Bays, neither SP itself cause a tension between Cantor's theorem and LST nor SP and LST themselves pose a problem for realist approach to set theory (Bay 2014). Because of this fact, Skolemite has to show step 1 of his argument by more broad constructive analysis of axiomatic approach rather than criticism of realist approach. Then step 2 and 3 can follow (Bay 2014).

2.4. Philosophical Bearings of Skolem Paradox: Hilary Putnam

Hilary Putnam offered another argument out of Skolem's arguments which is known as the model theoretic argument against realism. Putnam presented his formulation of SP and argued that it poses an antinomy not in formal logic but in philosophy of language (Putnam 1980, p.464).

Putnam's argument directed towards realistic semantics which argues that there exist certain objects such as sets and the universe of sets and words and sentences in the case of ordinary language, also symbols and well-formed formulas in the case of formal language refers and correspond to these objects in determined manner (Bays 2000, pp.79-80). Putnam claims that the language, ordinary or formal, doesn't refer or correspond to them determinately, namely it is semantically indeterminate (Bays 2014).

Putnam begins with three main positions on reference and truth (Putnam 1980, p.464). First of them is the extreme Platonist position. According to this position, human have nonnatural mental power which allows them to directly grasp forms. Putnam argues that although this position isn't subject to his general argument appealing to such mental powers isn't helpful as epistemology and convincing as science. Also there is role which some kind of rationality plays in the case of building mathematical axioms. But if this kind of rationality doesn't play this role in the case of formulating axiom of choice and the continuum hypothesis his argument also makes this position doubtful (Putnam 1980, p.471).

Second of these positions is the verificationist position. According to this position, in the case of understanding the language notion of proof or verification must be replaced with the classical notion of truth. Putnam argues that although it means rejecting the metaphysical realism this position is only position that isn't subject to his critism and retains empirical realism (Putnam 1980, p.464).

Third of these positions is the moderate realist position which is the subject to his critism and can't answer them. This position while doesn't accept nonnatural mental powers as first positions claims that classical notions of truth and reference is central (Putnam 1980, p.464).

According to Bays, the overall argument can be shown thusly:

> 1. Theoretical and operational constraints do not fix a unique "intended interpretation" for the language of set theory.
> 2. Nothing other than theoretical and operational constraints could fix a unique "intended interpretation" for the language of set theory.
> So, 3. There is no unique "intended interpretation" for the language of set theory (Bays 2000, p.84).

Theoretical constraints are ingredients of any formal system, formal set theory or formal total science (if it is possible to formalize totality of science), such as axioms, theorems included in them (Putnam 1980, p.466). Because of LST no theoretical constraints can rule out the "unintended interpretation" for the language of set theory. For this Putnam uses strong form of LST which states that if first order theory has any model it has a countable model which is submodel of first model. Since for any such formal theory if it is satisfiable there will be always another countable model in which name and predicate symbols refer relations, namely set of n-tuples, defined in it hence "uncountable", "finite", "infinite", "simply infinite sequence" will refer different relations (Putnam 1980, p.465). Thus set theoretic notions are semantically indeterminate.

Operational constraints are ingredients of empirical observations and measurements which are involved in science (Putnam 1980, p.469; Bay 2014). By LST any such physical science will fail to determine a unique intended interpretation for the symbols and formulas of set theory (Putnam 1980, p.466; Bays 2000, p.82).

Let it pose a problem for who wants to accept set theoretic reality. What about the mathematician who only wants which sentences of set theory are true? Since by each of these two different models the same sentences can be satisfied and turn out to be true. But the argument can be extended to include not just indeterminacy of semantics but also indeterminacy of truth value.

For this Putnam uses independence of an axiom which Gödel proposed it for set theory, namely the axiom V = L. V is the universe of all sets and L is the class of all constructible sets. So this axiom says that the universe of all sets is the class of all constructible sets. Independence of this axiom means that if ZFC is consistent thus both ZFC + V=L and ZFC + V≠L is consistent. Thus if ZFC has a model, there is a model which this axiom is true and there is a different model which this axiom is false. Since neither theoretical constraints nor operational constrainst can fix which of two models is the intended interpretation, the truth value of "V=L" is relative (Putnam 1980, pp.467-469). With this the first step of Putnam's argument is summed up.

"But if axioms cannot capture the intuitive notion of set, what possibly could? (Putnam 1980, p.465)" For the second step Putnam argues, as mentioned above, that falling back to the extereme Platonist position has no use for epistomelogical and scientific purposes. Also there is no good realist explanation for mathematical reference (Bays 2000, p.90). Even if one try to explain how to fix an unique intended interpretation this explanation can be seen as more theoretical constraints which are not enough to fix it (Bays 2000, p.97). Therefore the third step can be entailed from first and second step.

The model theoretic argument of Putnam has caused ongoing debate which includes reponses and counter responses for both technical side of the argument and philosophical side of it (Bays 2014).

2.5. Philosophical Bearings of Skolem Paradox: Wittgensteinian Position

Apart from these positions, in order to argue a version of the Skolemite position, Crispin Wright has appealed to Wittgenstein's thought regarding the relation of use to meaning, in addition to Kripke and Quine (Bays 2014; Wright 1985). Also Allen W. Moore has argued that if SP is considered in the context of the world and its limits as described in *Tractatus* it will appear as a genuine paradox (Moore 1985, p.14). Since the general idea of the Skolemite position is described above, Moore's Wittgensteinian Position will be presented here.

In *Tractatus* Wittgenstein has tried to expose the world and its limits. According to Wittgenstein combination of objects constitute states of affairs and existence of states of affairs forms facts. The world is basically the facts in logical space (Wittgenstein 2002, p.5). Because of the fact that "the limits of my language mean the limits of my world (Wittgenstein 2002, p.68)" while talking about the facts contained in the world is meaningful talking about its limits and about it will lead to meaninglessness (Moore 1985, p.14). According to Moore, LST will lead to such meaningless talk, which in turn will appear as paradox, when LST considered as it is about the world and its limits. Such a meaningless talk will appear as paradox because "contradiction is the outer limit of propositions (Wittgensteion 2002, p.48)." Thus "the debate between relativists and nonrelativists is destined to remain irresoluble (Moore 1985, p.14)."

Moore begins with the observation that although the terms such as "every set", "… is a member of _", "the hierarchy of sets" don't refer to any entity and make sense, "they seem to elucidate something which is apparent in (shown by?) the very fact that we can make generalizations about sets (Moore 1985, pp.15-16)." As mentioned above, because of LST, the set that satisfies the formula "there is an uncountable set" will appear from a point of view within the model as uncountable while from another point of view outside the model it will appear as countable. Moore focuces on such a presentation itself and argues that it is possible if the discourse about sets is limited to "a particular collection of sets, the collection to which such claims must be relativized (Moore 1985, p.18)." But this is possible if we accept that there is set of all set which the discourse is about. Since the claim that relativity of uncountability and absoluteness of countability entail that *all sets* are countable from an absolute point of view.

Entire debate between the relativist and the non-relativist concerning SP is about a limit of the world. There is no absolute point of view without accepting that the discourse about sets is about the entire hierarchy which is untenable (Moore 1985, p.19). Thus this debate "is in a very deep sense irresoluble (Moore 1985, p.19)."

2.6. The Need for Transcendental Exposition of Skolem Paradox

The arguments above seem to depend on two crucial ideas. On the one hand, they try to clear away the relations between object language and metalanguage and also between object language and its models, surrounding LST and SP, so as to show that if these relations are understood in a specific manner it will show that from LST and SP their consequence follows. On the other hand they argue that the universe of sets is unintelligible and there is no other way to understand the set theoretic notions than the axiomatic approach.

The first idea seems to assume that there is an objective formal language and it is constituted just by listing and enumerating its symbols with the help of recursive functions defined in syntactical manner. And only problem is that in what way the relations of this objective formal language to metalanguage and interpretations are need to be understood. Careful attention to this assumption will show that elements of formal language can't be constituted as a formal object just by listing.[3]

The second idea seems to assume that there is no intuition unless it is understood as sensual or some kind of ambiguous common sense. This assumption finds its roots in Kant's critique of reason.

In following chapters we examine these two ideas. In chapter 3 Kant's Idea of World will be described in relation to the constitution of object. As it will be seen Kant also thinks that Idea of World doesn't refer to an object and has no role in the constitution of object. In chapter 4 Cantor's diagonal argument will be presented. We argue that Cantor's representation of diagonal argument depends on some kind of rational idea which can be seen as the a priori condition of uncoutability's constitution. Then we examine consequences of this argument for SP.

[3] We present this claim in Chapter 4.

CHAPTER III

ON WORLD AS CONSTITUTED BY THE IDEA OF REASON

In this chapter, firstly, Kant's thought regarding the constitution of objects will be described briefly. Secondly connection of the World as the idea to the constitution of wholes made up of objects will be presented. Finally SP will be interpreted in regard to Kant's thought.

3.1. Constitution of Object

In *The Critique of Pure Reason,* Kant exposes the constitution of objects through experience so as to find *a priori* conditions of this constitution in pure reason. According to Kant, Object with its correspondant in intuition[4] is synthesized and comprehended by means of the spontaneous faculties. By this way, Object is comprehended in understanding from one aspect by means of concepts. Object which is comprehended from one aspect in this way is comprehended from different aspects in reasoning by means of transtition from concepts to anothers (Çitil 2012, pp.32-33).

As for the synthesis of object with its correspondant in intuition Kant argues that reception of representations by faculty of sensibility leads to cognition, namely whole of connected representations, if it is combained with synthesis of spontaneous faculties (Kant 1998, p.228). Such a synthesized, namely constituted, object by means of representations is comprehended in understanding by means of concepts that is related to synthesized representations (Kant 1998, p.205).

[4] In Kant's terminology object amounts to 'Objekt' and correspondant in intuition to 'Gegenstand'. This distiction is suggested by Çitil (Çitil 2012, p.23).

Faculty of sensibilty receives representations through affection from objects[5] (Kant 1998, p.155). These representations can not stand separated from each other and connection of them must be done in intuition before any other synthesis. Since they are in sensual intuition, they are subject to a priori forms of sensibility, i.e. time and space. Therefore they are connected to each other by means of relations of subordination and coordination. This connection of representations in intuiton is called synopsis and this act of synthesis is called apprehension (Kant 1998, pp.228-229).

So as to regard different representations as ascribable to one and same thing they must be reproduced in accordance with certain rules. Since representation of one and same thing is different in the course of time and, for example, even though it appears now as black and light it was heavy and white thus there are two appearances in two different moments, These moments and appearances must be reproduced as they are related to same thing even though this thing is not before sensibility. This reproduction must also be done in accordance with certain rules, not random so as to have a unity. In consequence the synopsis, also, must be subject to another synthesis. This synthesis is done by the faculty of imagination and this act of synthesis is called reproduction (Kant 1998, pp.229-230). And the rules which the reproduction is done in accordance with are pure schemata of imagination. They provide the application of categories and concepts of understanding to representations of senses (Kant 1998, p.257). They can provide this application since they are the transcendental determination of time, a priori form of inner sense (Kant 1998, p.272).

For this reproduction of representations to become cognition it must be subject to judgement. Every judgement is always a representation of one object (Kant 1998, p.205). But for the judgement to be about one object the consciousness of that what is thought in the course of time is the one and same is needed. "Without consciousness that which we think is the very same as we thought a moment before, all reproduction in the series of representations would be in vain (Kant 1998, p.230)." With this consciousness that unifies reproductions into one representation object can be represented in the judgement by means of concepts. On the other hand

[5] From 'objects' here Kant means 'thing in itself'. According to Kant, while this affection of sensibility from the thing in itself is thought it can't be known, i.e. cognized.

concepts are related to only to representations which are synthesized by means of apprehension and imagination, not directly to the object that affects sensual intuition. The spontaneous act which ascribes reproductions to consciousness of subject, namely 'I', and in turn unifies them into one representation is called apperception (Kant 1998, p.232). This synthesis is done in accordance with concepts. Therefore:

> the original and necessary consciousness of the identity of oneself is at the same time a consciousness of an equally necessary unity of the synthesis of all appearences in accordance with concepts, i.e., in accordance with rules that not only make necessarily reproducible, but also thereby determine an object for their intuition (Kant 1998, p.233).

Hence Kant determines the three sources on which the possibility of congnition of objects rest: sense, imagination and apperception (Kant 1998, p.236). The faculty of sensibility represents affections and appearances of thing in itself. Imagination apprehends and reproduces them. Finally apperception recognizes them in the consciousness "of the identiy of these reproductive representations with the appearances through which they are given (Kant 1998, p.236)." When it is considered in relation to the synthesis of imagination, the unity of apperception is the understanding (Kant 1998, p.238).

3.2. The Unity of Reason

Object which is comprehended from one aspect is subject to transition trough concepts by faculty of reason so as to be comprehended from different aspects. Reason does this transition in order to give a priori unity, in other words totality, through concepts to the rules of understanding without direct relation to experience or any object (Kant 1998, p.389).

According to Kant propositions such as "All humans are mortal" and inferences that don't rely on the mediation of third representation such as conclusion "Some humans are mortal" derived from "All humans are mortal" are thought through understanding (Kant 1998, pp.389-390). Reason by its natural propensity oversteps any given two representations and transcends them so as to find third representation (Kant 1998, p.591). This third representation can be in the manifold under higher genera or lower genera or intervening between two given representations (Kant 1998, p.598). Thus any inference that relies on the mediation of third represention, i.e. syllogism is thought through reason (Kant 1998, p.390). If such a representation can be found in the possible experience then conclusion of syllogism is valid.

Reason doesn't only transcend any given two representations but also it wants to go beyond every given representations so as to obtain the greatest unity and extension. Only by this way it can be thought that all the representations which have been given up to a point are only part of the whole of possible experience (Kant 1998, p.591). While such unity of reason is necessary for obtaining the greatest possible extension from this consideration there arises the irresistible deception as if this unity is obtained from an object which is outside of possible experience. But since reason doesn't relate to an object but only concepts by which understading unites representations into object and the relation of reason to concepts is the ordering relation so as to give them the unity thus this unity can not be obtained from an object (Kant 1998, pp.590-591).

On the other hand this unity of reason is obtained from an idea:

> This unity of reason always presupposes an idea, namely that of the form of a whole of cognition, which precedes the determinate cognition of the parts and contains the conditions for determining a priori the place of each part and its relation to the others. Accordingly, this idea postulates complete unity of the understanding's cognition, through which this cognition comes to be not merely a contingent aggregate but a system interconnected in accordance with necessary laws (Kant 1998, pp.591-592).

3.3. The Ideas of Reason

An Idea is a concept that has its origin solely in reason and it transcend the possility of experience (Kant 1998, p.399). For this definition to be understood clearly these should be noted. As mentioned above, time as a priori form of sensibility is one of the sources and conditions of possible experience. The pure schemata of imagination which bridges between concepts of understanding and representations of senses are the transcendental determinations of time. In regards to these two notes, transcending the possibility of experience means that Idea doesn't set time as condition for itself.

But because of this fact, on the other hand, it can't have any pure schemata in imagination which make possible the application of idea to representations of senses. Hence idea can't play any role in the synthesis of object, namely constitutive role in regards to object. But ideas determine the whole in which each part has the place a priori and each place have necessary relations to the others and ideas order concepts so as to take place in this whole. Thus ideas play regulative role bringing unity to the concepts of understanding. This is only projected unity, namely not given unity in itself but only a rule or principle which "help to find a principle for the manifold and particular uses of understanding, thereby guiding it even in those cases that are not given and making it coherently connected (Kant 1998, p.593)."

Since ideas transcend the possibility of experience they don't have any correspondant in intuition. Thus, according to Kant, ideas can be determined, not by transcendental exposition of experience, but by transcendental analysis of the form of syllogisms just as transcendental analysis of form of judgements showed the origin of pure concepts of understanding. This analysis is done in accordance with the function of reason in its inferences (Kant 1998, p.399). Reason seeks in its inferences totality of conditions for contidioned so as to determine a ground of synthesis for what is conditioned (Kant 1998, p.400). From 'conditioned' Kant means any object acquired from experience and comprehended by means of concepts through understanding such as "Caius is mortal". Condition of this proposition is human. Human is a "concept containing the condition under which the predicate (the assertion in general) of this judgement is given (Kant 1998, p.399)." According to Kant, Idea is the unconditioned according to which the totality of conditions is possible as if all concepts and rules of understanding containing conditions converge at pure concepts of reason, namely ideas, containing unconditioned. Since there are three relations of propositions in the forms of syllogisms, namely categorical, hypothetical, disjunctive thus, accordingly, there are three unconditioned concepts, namely the soul which is the thinking subject, the world which is the sum total of all appearances and the god which is the being of all beings (Kant 1998, p.406).

According to Kant, all three kinds of ideas are schemata of reason. They don't show how correpondants in intuititon of objects are constituted as in the case of schemata of imagination but they show how, under the guidance of these schemata, the constitution and connection of objects of experience in general must be sougth (Kant 1998, p.606). Kant also calls them maxims of reason since the understanding is obliged to obey them solely because of interests of reason (Kant 1998, p.603).

As for the idea of soul, following this idea, all representations and actions of inner experience are connected and converged at the soul as if it was a simple identical substance. Although, at first glance, for each effect there is a power to be found but one must assume that as if all variety of effects appeared in human mind were brought light by one absolutely fundamental power. This assumption is made, not in order to find in fact such an absolutely fundemantal power, but rather in order to seek systematic unity in cognition. Since there is no way that synthesis and comprehension for this power can be done. But, by this way, unanimity and relations of each power of mind, such as imagination, memory, consciousness etc., can be found even further than first glance (Kant 1998, p.594).

As for the idea of world, following this idea, “we have to pursue the conditions of the inner as well as the outer appearances of nature through an investigation that will nowhere be completed as if nature were infinite in itself (Kant 1998, p.606).” By this way whenever two concepts as in categorical judgments or two judgements as ground and consequence in the hypothetical judgements are thought togehter through understanding there can be found third representation as a condition of relation between two concepts or judgements. Following this way these conditions are connected and converged at the world as if it was an infinite complete totality of conditions. This totality will be examined further in a moment.

As for the idea of God, following this idea, all representations of both mind and nature are connected and converged necessarily as if they were caused by a highest intelligence (Kant 1998, p.607). By this way the understading’s concepts are connected systematically to each other with necessity. Kant also suggests that the idea of God comes before than other ideas and opens the field for other ideas (Kant 1998, p.614).

As a consequence of these three ideas the projected unity of reason as the form of whole of cognition is determined a priori. But since this unity is determined by ideas neither it can not be acquiered as an object nor it can be achieved as a goal. But all empirical investigations, in relation to these ideas, mean approximation to these unity and goal (Kant 1998, p.592).

3.4. On World as Regulative Idea of Reason

As mentioned above the unity of reason presupposes the formal whole. This whole comes before any of its determinate cognition of parts and this whole contains conditions of a priori determination of the places. In this whole there are determined places for each part and determined relations of a place to the others. These places are infinite, continuos and third-dimensional. It is the idea of world that furnishes this whole with these properties.

As for third-dimensionality it is the reason by the idea of world that allows the intention of the mind to turn towards and to seek third representation outside the two representations which are thought through understading. Whenever any two representations are given, idea of world ensures understanding that if it seeks third representation containing condition under which predication of one of them to the other or assertion of judgement in general is made possible then it can find it. The idea of world also ensures that whenever any two representations are given there will be third representation which is higher genera than these two and also another representation which is lower species than these two (Kant 1998, pp.596-598). It is the same case for any two judgements given as ground and consequence in the hypothetical judgement. This formal whole, in this sense, is third-dimensional as I call it. It should be noted that this property is ascribed to concepts comprehending objects, not to correspondents in intuition which are synthesized by spontaneous acts.

As for continuity and infinity, it is because of the reason with the help of the idea of world that all concepts, ascending to higher genera and also descending to lower species, are akin to one another. Kant calls this the affinity of all concepts (Kant 1998, p.598). According to this affinity, all concepts are bound to one another and in transitition from one to another there is no leap. Thus “from each one can reach another … intervening species are always possible, whose difference from the first and second species is smaller than their difference from each other (Kant 1998, p.599)." On the one hand, since according to their beings in nature all species are partitioned and thus have discrete quantum this affinity can not be evinced in experience. Otherwise between any two species there would be infinite intermediate members. On the other hand because of this affinity one can and must seek another species other than species that are already given (Kant 1998, p.600).

To sum up, the idea of world regulates the understanding so as to give all its concepts a determined place in the formal, third-dimensional, continuous and infinite whole. Hence the understaning can approximate from these concepts to the greatest unity and the highest degree of extension.

3.5. On the Universe of Sets as the Formal Whole of Reason

The Formal whole of reason is a projected unity in which there are determined places for each part and determined relations of a place to the others. There is always a third place intervening between any two places thus it is a third dimensional whole in this sense. There is no leap between transitions from one place to another. Intervening places are always possible. Thus it is a continuous whole and from continuity infinity follows. These three properties of the formal whole are provided as schemata of reason by the idea of world.

Following Kant, the universe of sets can be regarded as the formal whole of reason. While set theorists such as Freankel saw the certain kinship between paradoxes of the universe of sets and the antinomies of reason (Hallett 1984, pp.224-225) and this kinship is investigated by Michael Hallett (Hallett 1984), there is no investigation as regards to relation of the universe of sets to the formal whole of reason. Allen W. Moore just suggested that the universe of sets can be seen as idea of reason but do not deepen this suggestion (Moore 2001, p.171).

Accordingly the idea of the universe of sets will allow the discourse of sets to be about all sets and each set will take place in this formal, continuous and infinite universe. The universe of sets itself can't be regarded as a set or as an existing object. Since it is only a projected unity, regading it as an object would lead to antinomies and paradoxes. But because of this unity, one can always hope to find and in fact find another set, by ‘forcing’ or another method, other than the sets which have already given. Thus this formal whole is indispensable to the discourse of sets. On the other hand the given sets, in any moment of the investigation, have, not continuous, but discrete quantum.

Continuing this reasoning there are two points of views. From the point of view of reason there is the universe of sets. From the point of view of understanding there are only given sets. Thus latter point of view is not absolute one as Skolemites argue but the former one is the absolute point of view and on the contrary of Putnam there is the absolute point of view from which one can determine the intended model.

According to these considerations LST, downward version, can be interpreted as the following: From the absolute point of view for any first-order formal axiomatic language which is consistent, there is a model. But if this absolute point of view is disregarded and only the sets which can be properly ‘synthesized’ and given are regarded there is a countable submodel of the first model for this formal language.

According to this interpretation SP appears if this disregarding of the absolute point of view is forgotten. Without this SP don’t pose any legitimate philosophical objection for uncountable domain. It is necessary for the discourse of sets as the formal whole of reason.

But it should be noted that even with these interpretations one can’t claim that there is an uncountable set as an object. To this point the uncountable set is, although it is necessary, merely a projected unity. In the next chapter we try to show that the schemata of three dimensionality and continuity are presupposed in the representation of uncountability. In doing so we claim that there is an idea underlying the constitution of the uncountability. This claim will evince that Kant’s thought as regards to ideas of reason is in need of revision.

CHAPTER IV

EXPOSITION OF CANTOR'S DIAGONAL ARGUMENT

In this chapter, firstly, we try to describe that the representation of a formal theory presupposes both the places in which they are situated and their ordering. Secondly we demonstrate Cantor's diagonal argument in two different ways as geometrical and set theoretic. Thirdly I'll discuss that the schemata of continuity and three dimensionality play a role in the representation of Cantor's diagonal argument. Finally we make some concluding remarks.

4.1. The Representation of Formal Theory

Defining a formal theory begins with the rigorously defining its formal language. In it there are symbols and formulas. Symbols are primitive signs. Formulas are strings of primitive signs that are assembled according to well-defined recursive rules. Such a formula is called well-formed formula or wff for short. After defining the formal language, deductive apparatus is defined so as to form a definite formal theory. Deductive apparatus is defined by determining certain formulas as axioms and rules of inference which allows deduction of other formulas, i.e., theorems from these formulas, i.e. axioms. With the help of deductive apparatus proofs are defined. Proofs are strings of wffs that are assembled according to rules of inference in the deductive apparatus. Thus the objects of the formal theory are symbols, formulas and proofs. In other words, in formal theory, there are primitive signs, strings of these signs and strings of these strings.

In order to exemplify above mentioned definitions, the formal language for propositional logic can be defined as following:

Primitive Signs:

- Connectives: $\wedge$, $\vee$, $\rightarrow$, $\neg$

- Sentence letters: p, q, r with numerical subscripts
- Parentheses: (,)

Definition of Well-Formed Formulas:

i) Every sentence letter is a propositional logic well-formed formula.

ii) If α and β are propositional logic wffs then $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, $\neg\alpha$ are also propositional logic wffs.

iii) Only wffs of propositional logic are the formulas which are determined by using i and ii.

With this, a formal language of propositional logic is defined. For example, whenever "p" is written down, it will be wff of propositional logic by rule i. From this wff, "¬p" can be formed by rule ii and also from "p" and "q", "(p ∧ q)" can be formed by same rule. Rule iii prevents any other formation of primitive sings to be regarded as belonging to propositional logic such as "(¬p¬p)".

To obtain a formal theory an addition of a deductive apparatus is needed. Deductive apparatus won't be defined completely here. Only one of the inference rules of the deductive apparatus will defined so as to show what proof looks like. These rules allow transition from a wff or wffs to another wff. One of such rules can be defined as below:

The rule of elimination of →:

From wffs $(\alpha \rightarrow \beta)$ and α, the transition to β is allowed.

According to this rule, if there are two wffs such as "((p ∧ q) → r)" and "(p ∧ q)" then from these two wffs transition to "r" is allowed. These three wffs taken together are called a proof in this theory. It should be noted that whenever the deductive apparatus is defined all rules and definitions of notions such as proof are as well defined.

It should be clear that any formula of the formal theory is a string of primitive signs in a definite order and any proof of the formal theory is a string of formulas in a definite order. Without ordering the primitive signs in a definite manner wffs and proofs can't be shown. With primitive sings these are the objects of the formal theory. In these definitions, according to Gödel, they are tackled with according to their 'outward appearance'. According to their outward appearance they have two main properties: the number of signs in them and the order of these signs (Çitil 1994, p.48).

Now the present thesis' question that is put forward in Chapter 2 was whether an objective formal language is constituted just by listing and enumerating its symbols with the help of recursive functions defined in syntactical manner. With above mentioned considerations, this question can be rephrased as follows: "whether a formal object rests ontologically on its representation or whether we can indentify a formal object with one of its representations (Çitil 1994, p.46)." In other words do these sings as ink blots written on a piece of paper or as pixels typed in a computer or as sounds said out loud stand on their own as a formal objects?

It appears that not only is there neglect of such an investigation but there is also tendency towards identifying the representation of formal object with itself. For present thesis' purposes this kind of investigation won't be detailed.[6] What it will be indicated is that there is more to formal objects than their representations as ink blot, pixels or sounds.

Let "$((p \wedge q) \rightarrow r)$" and "$(p \wedge q)$" be two wffs in the proof of the formal theory. "p" is also a wff. "p" occurs both in the former and the latter wff. Thus it assumed that "p" is the same wff that occurs in both wffs. But the representations of "p" are always different. If it is written down on paper twice, these representitions will have different inks, from the physical perspective different atoms, probably different geometrical shapes. Thus these two representations are basically two different objects. The same considerations apply if the wffs are said out loud with sound waves or typed on a computer screen. So the identity of elements of wffs doesn't come from such representations.

[6] Çitil has carried out a detailed investigation about the mentioned questions and more (Çitil, 1994).

Also while written or typed representations stand side by side in two dimensional space, wffs doesn't stand side by side. For example, in this wff "$((p \wedge q) \rightarrow r)$", "p" comes before "∧" and "q" comes after "∧". They are ordered one after another. It is the similar case if they are said out loud. In this wff "$((p \wedge q) \rightarrow r)$" sings ordered as the first sign, the second sign etc. Firstness, secondness and so on don't come from representations themselves (Çitil 1994, pp.53-58).

Also since signs such as "p", "r" (…) are arbitrary and one can replace any signs with others. Therefore what is ordered is not the signs themselves but the places which they fill in (Çitil 2012, p.158). When the multitude of places are identified and ordered one can write down any wff or any proof in them.

In conclusion a formal object is a set of signs that have identity and are situated in ordered places.

4.2. Cantor's Diagonal Argument

In 1891 Georg Cantor published an article entitled "On an elementary question in the theory of manifolds". In this article Cantor announced the method called diagonalization. With help of this method Cantor proved that there are uncoubtable sets. Cantor as well proved that for any set S, the cardinality of the power set of S is greater than the cardinality of S (Ewald 2005, p.920). According to Cantor, $\mathbb{R}$ and also a proper subset of $\mathbb{R}$ are examples of uncountable sets. The present thesis' focus is on the diagonalization and its use for the proof of uncountability of $\mathbb{R}$.

As for diagonal argument Cantor's steps can be shown as follow. Let m and w be two distinct characters and M be a set that has elements of the following form:

$$E = (x_1, x_2, x_3, \ldots , x_v, \ldots)$$

E_n is a set which has infinite elements and each of its elements is either m or w. Some elements of M can be listed as the following:

$$E^{I} = (m, m, m, \ldots)$$

$$E^{II} = (w, w, w, \ldots)$$

$$E^{III} = (m, w, m, \ldots)$$

Now, if it is tried to put elements of M into one-to-one correspodance with $\mathbb{N}$ so as to list and thus count all elements of M, this list L cannot be complete, namely there will always be an element of M which isn't among those listed in L. In other words "if E_1, E_2, E_3, …, E_v, ... is any simply infinite sequence of elements of the manifold M, then there is always an element E_0 of M which corresponds to no E_v (Cantor 1891, p.921)." It is the same case with the new list L* to which E_0 is added. There will be a new element E_0^* of M which corresponds to no E_v in this new augmented list L* (Boolos & Burgess & Jeffrey 2007, p.16).

The method is this. Confronted with such an infinite list L

$E_1 = (\mathbf{a_{1,1}}, a_{1,2}, \ldots, a_{1,v}, \ldots)$,

$E_2 = (a_{2,1}, \mathbf{a_{2,2}}, \ldots, a_{2,v}, \ldots)$,

…

$E_u = (a_{u,1}, a_{u,2}, \ldots, a_{u,v}, \ldots)$

…

of elements E of M, E_0 is defined as follows:

$E_0 = (b_1, b_2, \ldots, b_v, \ldots)$.

such that b_v is either m or w and it is different from $a_{v,v}$. For example if integral value of v is 1 and $a_{1,1}$ is m then b_1 is w. If integral value of v is 2 and $a_{2,2}$ is w then b_2 is m. Thus b_v always takes different values from the entries in the diagonal (upper left to lower right) array of the list and the set E_0 which is composed of diagonal sequences of b_v is called the diagonal set. It is clear that this defines a set. It is also clear that this set is an element of M since it satisfies the definition of being an element of M (Boolos & Burgess & Jeffrey 2007, p.16).

Suppose that there is an element E_u which is listed in L. If $E_u = E_0$ then each integral value of v, $b_v = a_{u,v}$. But this will entail that for the integral value u of v, $b_u = a_{u,u}$. This is a contradiction and E_0 isn't listed in L (Cantor 1891, p.921).

As a consequence "the totality of elements of M cannot be brought into the sequential form: {E_1, E_2, E_3, …, E_v, ...}. Otherwise, we would have the contradiction that a thing E_0 would be an element of M as well as not an element of M (Cantor 1891, p.921)."

As for the proof of uncountability of $\mathbb{R}$, the presentantion below follows the presentation of Yaqup (Yaqup 2015, pp.106-107) with some revisions. It is quite similar to Cantor's presentation. But since Cantor's presentation is oppressed (Cantor 1891, p.922), the former presentation is preferred.

The reductio ad absurdum proof goes like this: Let M be the set which is composed of all real numbers that are greater than or equal to 0 and less than 1. To be more precise, $M = \{x: x \in \mathbb{R} \text{ and } 0 \leq x < 1\}$. Let r_n be the elements of M and have the following form:

$$r_n = 0.a_0^n\ a_1^n \ldots$$

The sign n represents the i^th real number in M and the sign a can be any digit from 0 to 9. For example first real number in M can be represented thusly: $r_0 = 0.a_0^0\ a_1^0\ a_2^0 \ldots$ and each a is 0, namely $r_0 = 0.000\ldots$.

It is clear that M is a proper subset of $\mathbb{R}$. If it can be shown that the cardinality of M is greater than the cardinality of $\mathbb{N}$, then it implies that the cardinality of $\mathbb{R}$ is greater than the cardinality of $\mathbb{N}$ as well.

In this proof, the real numbers is thought in terms of their infinite decimal expansions. To prevent the same real number from occurring in the list twice, the decimal expensions ending with infinite sequences of 9's are extracted from the list.

Now, assume that the cardinality of M is equal to the cardinality of $\mathbb{N}$, namely they can be put one-to-one correspondence. Thus the members of M can be put in an infinite list in which each and every member of M appears once and only once. Here is how this list looks:

$$r_0 = 0.a_0^0\ a_1^0\ a_2^0\ a_3^0\ a_4^0\ a_5^0 \ldots$$
$$r_1 = 0.a_0^1\ a_1^1\ a_2^1\ a_3^1\ a_4^1\ a_5^1 \ldots$$
$$r_2 = 0.a_0^2\ a_1^2\ a_2^2\ a_3^2\ a_4^2\ a_5^2 \ldots$$
$$r_3 = 0.a_0^3\ a_1^3\ a_2^3\ a_3^3\ a_4^3\ a_5^3 \ldots$$

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Confronted with such an infinite list, let d be the real number of the following form:

$$d = 0.b_0\ b_1\ b_2\ b_3\ b_4\ b_5\ \ldots\ b_v\ \ldots$$

such that b_v is one of the digits from 0 to 9 and if $a_v^v = x$ then $b_v = x+1$ and if $a_v^v = 9$ then b_v is 0. For example $a_0^0 = 0$ then $b_0 = 1$ and if $a_1^1 = 2$ then $b_1 = 3$ and if $a_5^5 = 9$ then $b_5 = 0$ and so on. d is clearly a member of $\mathbb{R}$ and it is between 0 and 1.

It is assumed that the members of M can be put in an infinite list. Thus d must be somewhere in the list. Using the notation of list, d can be represented as following:

$$r_k = 0.a_0^k\ a_1^k\ a_2^k\ a_3^k\ a_4^k\ a_5^k\ \ldots\ a_k^k\ \ldots$$

such that $a_k^k = b_k$.

But by definition of d, if $a_k^k = x$ then $b_k = x+1$ and if $a_k^k = 9$ then $b_k = 0$. Thus $a_k^k \neq b_k$. This is a contradiction and the reduction assumptio must be false.

There is and there will be always a real number which doesn't occur in the list. Thus the members of M can't be put one-to-one correspondace with $\mathbb{N}$ and the cardinality of M isn't equal to the cardinality of $\mathbb{N}$, but is greater than that of $\mathbb{N}$. As consequence, $\mathbb{R}$ is an uncountable set.

Considering countability of the set of rational numbers, $\mathbb{Q}$, the uncountability of $\mathbb{R}$ is distinctive property of $\mathbb{R}$. Thus while $\mathbb{N}$ and $\mathbb{Q}$ can be brought into the sequential form,

$$E_1, E_2, E_3, \ldots, E_v, \ldots$$

$\mathbb{R}$ can not be brought into it.

It is the similar case with the power set of any countably infinite set such that for any countably infinite set S, the cardinality of the power set of S is greater than the cardinality of S (Cantor 1891, p.922).

Cantor ends his article with this observation:

> … in Nature there is for every power a next greater, and moreover every infinite ascending set of powers is followed by a next-greater.
> The 'powers' represent the unique and necessary generalization of the finite 'cardinal numbers'. They are none other than the actual-infinite cardinal

> numbers, and they have the same reality and determinateness as the others (Cantor 1891, p.922)[7].

Thus, according to Cantor, the universe of sets that consists of the power sets of infinite sets is the actual-inifinite. Although it can not be represented mathematically it is the representation of the Absolute (Cantor 1883, p.891).

4.3. Exposition of Cantor's Diagonal Argument

As mentioned above a formal object is a set of signs that have identity and occur in ordered places. The sequential form

$E_1, E_2, E_3, \ldots, E_v, \ldots$

doesn't consist of ink blots written or pixels typed but set of signs as such. It is a formal object and the signs used there occur in ordered places. These places must be recognized before any sings so as to constitute a formal object.

$\mathbb{N}$ and $\mathbb{Q}$ can be brought into this form. This means that the digits that represent elements of $\mathbb{N}$ and $\mathbb{Q}$, can be put in the same kind of ordered places and take their order. Thus the representation of the form

$1, 2, 3, 4 \ldots, n, \ldots$

can truly represent $\mathbb{N}$. In this form digits or numerical signs such as 1, 2 and so on occupy a place cognized before them and take their order. These digits do not 'read' from 'left' to 'right' but from first sign to second sign and so on. This can be clearly seen if it is considered that these signs can be written vertically from up to down or down to up but still have the order. Also with enough attention it can clearly be seen that the places that formal objects occupy are not the places that physical ink blots or pixels take. For example if we write down the word "apple" on a paper each ink blots occupy a physical place on the paper. But "a" written on the paper isn't same "a" that is a formal object. Because of the fact that I can write down "elma" on the same paper and these two a's will be different ink blots as physical objects but they are exemplification of the same formal object "a".

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[7] Cantor's views about actual infinity and the absolute are open to debate. We do not focus on these discussions in the present thesis.

As it is shown, $\mathbb{R}$ can not be brought in the sequential form above. This means that the representation of the form

$r_1, r_2, r_3 \ldots, r_n, \ldots$

such that each sign represents an element of $\mathbb{R}$, can not truly represent $\mathbb{R}$. There is always another sigh that represents an element of $\mathbb{R}$, let it be "d", and it can not be put in this form.

This is because of the fact that the formal places of the sequential form aren't convenient 'places' for the signs that represent elements of $\mathbb{R}$. The sign "d", although it appears that it should be in this form, can not occupy a place in it. Its 'place' is anywhere between two places that come one-after-another. Whenever any two places which formal objects occupy are thought there is always a third 'place' intervening between these two places. It is a margin that separates these two places and in this margin the sign "d" take place. Contrary to the places margins can never be filled. If d is added to the sequential form so as to rectify the defect, there would be another sign d^*, although it appears that it occupies a margin in this form, can not occupy a place in it. It should be noted that the representations of $\mathbb{Q}$ occupy the same place as that of $\mathbb{N}$, since $\mathbb{Q}$ is a countable set.

Since the places of signs must be cognized before signs, margins are cognized before any representations of elements of $\mathbb{R}$. Thus the representation of the diagonal argument presupposes the cognition of both places and margins.

To chew the term "margin" finer, margin is a term labeled according to outward appearance of sings. According to constitution of formal objects, margin is a gap between any two places. If it is considered that each place which formal objects fill in is unique and has particular unity these places can be considered as unique points. It isn't possible to claim that any two unique points are differentiated via some another point, because this claim has to answer what does differentiate these three points and so on? Thus it will lead to infinite regress. As a consequence any two unique points are differentiated via some thing of another kind, namely a gap (Çitil 2012, pp.103-104).

To make it clearer, let us consider the cognition of the two unique points, A and B. This cognition includes three judgements: “There is A”, “there is B” and “The points A and B are differentiated by gap”. The notion of gap can be rephrased as the judgement: “A isn’t B and B isn’t A”. From this consideration it should be clear that what we call “gap” as regards to formal objects an only be presented as a contradiction at the level of discursive thought (Çitil 2012, pp.104-105).

Now, it can be seen that the places and margins together forms a third dimensional formal whole in the sense defined in Chapter 3. Since whenever any two places are given there will always be third ‘place’ that intervenes between these two. It is the cognition of margin that makes it possible such a cognition for the understanding. This whole is also continuous. Since there is no leap between any two places and intervening places are always possible.

To represent the real number that doesn’t appear in the list, one can also use set-theoretic representation. But this representation also presupposes three dimensionality since in order to define “d” one has to use tuples that have three elements, namely triple. Consider the list in 4.2 and let F be function that takes the digits that the j^{th} decimal digit (which is after “0,”) of the i^{th} real number in the list and if it is x then alters it to x+1. Then the set that consists of triples may looks like this: {(1,1,2), (2,3,4), (3,5,6), …}. But this also is possible if one uses threefold representations and it relies on three dimensionality.

As a consequence, if what is told up to this point is true then it is shown that the schemata of three dimensionality and continuity are presupposed in the representation of uncountability. Also these two schemata play role in the constitution of formal objects.

4.4. Concluding Remarks

With these considerations, it should be said that the schemata of three dimensionality and continuity have constitutive role in both the universe of sets and the representations of it. Although Cantor showed that there is an uncountably infinite set by reductio ad absurdum, there must be such an idea that makes possible the cognition of uncountable sets beforehand. If this idea, which provides these two schemata, plays a constitutive role then the universe of sets and the notion of uncountability can't be disregarded. Both the universe of sets and the notion of uncountability have transcendental source in pure reason which makes them objective.

This kind of thought is also analogous with that of Cantor:

> The words 'finite understanding' which one hears so frequently are in my opinion not at all apt; however limited in truth human nature may be, still very much of the infinite adheres to it, and I even assert that if it were not itself in many respects infinite, the solid confidence and certainty in the being [Sein] of the Absolute, about which we know we all agree, would be inexplicable (Cantor 1883, p.891).

Considering the universe of sets and the notion of uncountability in relation to an idea of pure reason, it can be said in Kantian terms that the infinite adheres to human nature through reason and the finite adheres to it through understanding.

As Putnam pointed out, if one couldn't determine intended model of formal system, truth and existence would be relative. But since there are objective source of uncountable sets, one can determine the intended model and thus truth and existence isn't relative. Thus there is a relation of uncountability to the truth and existence. This relation can be seen also in the metalogical theorem which states that "there are uncountably many truths of the full theory of the natural numbers (Hunter 1971, p.29)." Since the idea of pure reason plays role in the constitution of objects there is a relation of this idea to truth. Thus it can be stated that the logical space of truths is uncountably infinite.

As for LST, This means that from an absolute point of view for any first-order formal axiomatic language one can see that there is an uncountable set. But from axiomatic point of view there is a countale submodel of the first model for this formal language as it is claimed in Chapter 3, as if one can talk about the Absolute from axiomatic point of view with deficiency. This deficiency is rooted in the representation of language. SP only shows this deficiency, not the relativity of uncountability or non-existence of it.

CHAPTER V

CONCLUSION

In this concluding chapter a summary of what is told up to now is presented and some questions that have not been discussed yet are briefly considered.

The logico-historical presentation showed that there is a tendency towards axiomatization of the notion of set and set theory because of the fact that the naive approach towards them has led to paradoxes. The justification of sole axiomatic approach comes from the fact that there is no other legitimate approach. But sole axiomatic approach come to a point which argues that all logico-mathematical notions can only hold in merely verbal sense. In addition to that truth and existence as they are used in the axiomatic systems come out to be relative.

We claimed that there is no sufficent investigation about both paradoxes of set theory and the formal language itself. We tried to show that transcendental investigation can show that paradoxes can be interpreted in a way that they don't lead to the non-existence of objects of mathematical notions. But they come to light only if one doesn't regard different transcendental sources of these notions. If these sources are considered, the theorems which seem to lead to paradoxes acquire entirely different meanings. In view of these considerations we tried to interpret LST and SP.

As the title of present thesis indicates, this is only an introduction to such an investigation. There are questions to be answered so as to complete the transcedental exposition. These questions can be enumerated as below:

If there is an idea that makes possible the constitution of objects, what is the exact relation of ideas of reason to these objects? The answer to this question will lead to a reconsideration of Kantian exposition of objects.

What is this idea exactly? I only indicate that there must be an idea that makes us possible to cognize the universe of sets in objective manner. But this idea remained vague. One must clarify this idea so as to complete this investigation. But if it can be clarifed, I believe that one can show analytical proof of the uncountability of $\mathbb{R}$ in contrary to Cantor's reductio ad absurdum proof. According to this analytical proof one can start from the idea and through analysis of this idea one can show that the uncountability of $\mathbb{R}$ follows. I believe that such a proof will pose an answer to Putnam's doubt about usefulness of the appeal to extreme Platonism.

Another question to be answered is this: what is the exact relation of formal language to sets? It appears that there is a deficieny in the relation of formal language to mathematical objects such as sets. What is the nature of this relation?

We claimed in our thesis that from an absolute perspective one can determine the intended model. If above mentioned questions are answered, these answer might shed light on the relation of truth and existence to the idea, the notion of uncountability and formal language.

We leave these questions to the further investigations. Let Nazım Hikmet (2010, p.1689), Turkish poet, have the last word:

> Oldum yıldızlarla haşır neşir
> ama sayısı bir tamam sayılamadı.

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